

Symmetry Classes of Alternating Sign Matrices

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Abstract

An alternating sign matrix is a square matrix satisfying (i) all entries are equal to 1, -1 or 0; (ii) every row and column has sum 1; (iii) in every row and column the non-zero entries alternate in sign. The 8-element group of symmetries of the square acts in an obvious way on square matrices. For any subgroup of the group of symmetries of the square we may consider the subset of matrices invariant under elements of this subgroup. There are 8 conjugacy classes of these subgroups giving rise to 8 symmetry classes of matrices. R. P. Stanley suggested the study of those alternating sign matrices in each of these symmetry classes. We have found evidence suggesting that for six of the symmetry classes there exist simple product formulas for the number of alternating sign matrices in the class. Moreover the factorizations of certain of their generating functions point to rather startling connections between several of the symmetry classes and cyclically symmetric plane partitions.

1 Introduction

An alternating sign matrix is a square matrix satisfying

- (i) all entries are equal to 1, -1 or 0;
- (ii) all rows and columns have sum 1;
- (iii) in every row and column the non-zero entries alternate in sign.

When we have occasion to consider the entries of an n by n alternating sign matrix, we shall always index these as a_{ij} where i and j vary from 0 to $n - 1$.

Alternating sign matrices have been studied¹ in [3], [4], [5], [6], and [7]. A much more recent and up to date account is given in David Bressoud's book [2]. Alternating sign matrices have been the source of a large number of conjectures related to their enumeration. In [4] and again in [5] there occurred discussions of alternating sign matrices subject to certain symmetry conditions. R. P. Stanley suggested putting this study in a more systematic framework in analogy with similar results described in [8] on symmetry classes of plane partitions.

The group of symmetries of the square acts in an obvious way on square alternating sign matrices. For each subgroup of the group of symmetries of the square one may consider the set of alternating sign matrices invariant under this subgroup. We call these sets the *symmetry classes* of alternating sign matrices. This paper is primarily a list of conjectures related to these symmetry classes.

There are 8 conjugacy classes of subgroups of the group of symmetries of the square. These give rise to 8 symmetry classes of alternating sign matrices. Several of the symmetry classes have been previously studied, but we repeat the results here for completeness. We describe the symmetry classes next.

¹I am posting this article at the request of Greg Kuperberg. It is essentially the same manuscript that I wrote sometime in the late 1980's. There has been much progress since that time particularly by Kuperberg and Doron Zeilberger, but, except for indicating reference [2], I have made no attempt here to bring the references up to date.

type	condition	
1		no conditions
2	$a_{ij} = a_{i,n-1-j}$	vertical axis
3	$a_{ij} = a_{n-1-i,n-1-j}$	half turn
4	$a_{ij} = a_{ji}$	diagonal
5	$a_{ij} = a_{j,n-1-i}$	quarter turn
6	$a_{ij} = a_{i,n-1-j} = a_{n-1-i,j}$	horizontal and vertical
7	$a_{ij} = a_{ji} = a_{n-1-j,n-1-i}$	both diagonals
8	$a_{ij} = a_{ji} = a_{i,n-1-j}$	all symmetries

We have computed the number of alternating sign matrices in each of the symmetry classes for small matrices. The numbers are shown in the first table of Section 4.

In many cases we can conjecture simple formulas expressing the number of n by n alternating sign matrices in a symmetry class as a product of small integers. These formulas are given in the second table of Section 4.

Our remaining results are concerned with generating functions for these symmetry classes. They point to rather startling connections between several of the symmetry classes and cyclically symmetric plane partitions.

2 Some Known Generating Functions

Many of the generating functions for symmetry classes of alternating sign matrices can be expressed in terms of three functions, $Z_n(x, y, \mu)$, $T_n(x, \mu)$ and $R_n(x, \mu)$, which we describe below and which have been proved elsewhere to have some interesting properties.

Let μ be a non-negative integer and define

$$Z_n(x, y, \mu) = \det(\delta_{ij} + z_{ij})_{0 \leq i, j < n}$$

where

$$z_{ij} = \sum_{t,k=0}^{n-1} \binom{i+\mu}{t} \binom{k}{t} \binom{j-k+\mu-1}{j-k} x^{k-t}$$

for $0 \leq i < n-1$, $0 \leq j < n$ and

$$z_{n-1,j} = \sum_{t,k,l=0}^{n-1} \binom{n-2+\mu-l}{t-l} \binom{k}{t} \binom{j-k+\mu-1}{j-k} x^{k-t} y^{l+1}$$

for $0 \leq j < n$.

We assign to $Z_0(x, y, \mu)$ the conventional value 1. When regarded as a polynomial in y , its coefficients are known to form a palindromic sequence. The cases $\mu = 0$ and $\mu = 1$ will be most important for us, so we list several of the values of $Z_n(x, y, 0)$ and $Z_n(x, y, 1)$ in the last table in Section 4.

Let

$$T_n(x, \mu) = \det \left(\sum_{t=0}^{2n-2} \binom{i+\mu}{t-i} \binom{j}{2j-t} x^{2j-t} \right)_{0 \leq i, j < n}.$$

The values of $T_n(x, 0)$ and $T_n(x, 1)$ will be particularly important for us. Several are tabulated in the last section.

Let

$$Y(i, t, \mu) = \binom{i+\mu}{2i+1+\mu-t} + \binom{i+1+\mu}{2i+1+\mu-t}$$

and define $R_0(x, \mu) = 1$ and

$$R_n(x, \mu) = \det \left(\sum_{t=0}^{2n-1} Y(i, t, \mu) Y(j, t, 0) x^{2j+1-t} \right)_{i, j=0, \dots, n-1}$$

for $n \geq 1$. Some of the polynomials $R_n(x, 0)$ and $R_n(x, 1)$ are given in the last section of this paper.

The first two functions are known to be the generating functions for classes of combinatorial objects.

Let $\mathcal{Z}_n(\mu)$ be the set of shifted plane partitions (arrays of positive integers)

$$\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \dots & & a_{1, \lambda_1} \\ & a_{22} & a_{23} & \dots & & a_{2, \lambda_2} \\ & & \vdots & & & \\ & & & & a_{rr} & a_{r, \lambda_r} \end{array}$$

with $\lambda_1 > \dots > \lambda_r$, with strictly decreasing columns and weakly decreasing rows, with no row length exceeding n and such that the first entry a_{ii} of each row exceeds the length $\lambda_i - i + 1$ of that row by precisely 2μ . We say that a part a_{ij} of such a plane partition is *special* if $\mu < a_{ij} \leq j - i + \mu$. To such a plane partition we assign a weight of $x^r y^s$ if there are r special parts and s parts in the first row equal to $n + 2\mu$. Then one may verify, by arguments like those in [6], that the sum of the weights of all partitions in $\mathcal{Z}_n(\mu)$ is the polynomial $Z_n(x, y, \mu)$.

We also remark that $\mathcal{Z}_n(0)$ is known to be in one-to-one correspondence with the class of cyclically symmetric plane partitions whose Ferrers graphs are contained in the box

$$\mathcal{X}_n = [1, n] \times [1, n] \times [1, n].$$

$\mathcal{Z}_n(1)$ is known to be in one-to-one correspondence with the class of all “descending plane partitions” with no parts exceeding $n + 1$ (which are conjectured to be in one-to-one correspondence with $n + 1$ by $n + 1$ alternating sign matrices.)

Let $\mathcal{T}_n(\mu)$ be the set of triangular arrays of positive integers

$$\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1,n-1} \\ a_{21} & a_{22} & \dots & a_{2,n-2} \\ \vdots & \vdots & & \\ a_{n-1,1} & & & \end{array}$$

such that all rows and columns are weakly decreasing and $a_{i1} \leq n - i + 1 + \mu$ for $i = 1, \dots, n - 1$. We say that a part a_{ij} of such a plane partition is *special* if $a_{ij} \leq j$. If such a partition has r special parts, we assign it a weight of x^r . Then one may verify that the generating function for $\mathcal{T}_n(\mu)$ is precisely $T_n(x, \mu)$.

We remark that $\mathcal{T}_n(\mu)$ is known to be in one-to-one correspondence with the set of partitions in $\mathcal{Z}_n(\mu)$ which are invariant under a certain involution. In particular when $\mu = 0$ these are in one-to-one correspondence with cyclically symmetric plane partitions which are equal to their transpose-complements. (see [6] for definitions). When $\mu = 1$, these are in one-to-one correspondence with descending plane partitions invariant under the involution described in [4]. The argument is sketched in [6]. The involution can be used to prove that in $Z_n(x, y, \mu)$ has a palindromic coefficient sequence when regarded as a polynomial in y .

Some other properties of the polynomials $Z_n(x, y, \mu)$ and $T_n(x, \mu)$ are known.

Theorem 2.1 *Let n be a non-negative integer. Then*

$$\begin{aligned} Z_{2n}(x, 1, \mu) &= T_n(x, \mu)R_n(x, \mu) \\ \text{and } Z_{2n+1}(x, 1, \mu) &= 2T_{n+1}(x, \mu)R_n(x, \mu). \end{aligned}$$

This is proved in [6].

Below we use the abbreviation

$$(X)_j = X(X+1)(X+2)\cdots(X+j-1).$$

The hardest result is due to Andrews [1]. We repeat its statement here.

Theorem 2.2 (Andrews) *Let*

$$\begin{aligned}\Delta_0(\mu) &= 2, \\ \Delta_{2j}(\mu) &= \frac{(\mu + 2j + 2)_j (\frac{\mu}{2} + 2j + \frac{3}{2})_{j-1}}{(j)_j (\frac{\mu}{2} + j + \frac{3}{2})_{j-1}}, \quad j > 0, \\ \Delta_{2j-1}(\mu) &= \frac{(\mu + 2j)_{j-1} (\frac{\mu}{2} + 2j + \frac{1}{2})_j}{(j)_j (\frac{\mu}{2} + j + \frac{1}{2})_{j-1}}, \quad j > 0.\end{aligned}$$

Then

$$Z_n(1, 1, \mu) = \prod_{k=0}^{n-1} \Delta_k(2\mu).$$

The proof in [1] is difficult. An easier proof which applies only to $\mu = 0$ and $\mu = 1$ is given in [3].

Using Theorems 2.1 and 2.2, one can prove a similar formula for $T_n(1, \mu)$.

Theorem 2.3

$$T_n(1, \mu) = \frac{1}{2^n} \prod_{k=0}^{n-1} \Delta_{2k}(2\mu).$$

A proof is given in [6].

Note that a consequence of Theorems 2.2 and 2.3 is that, for fixed integral μ , when we substitute $x = 1$ and $y = 1$ in $Z_n(x, y, \mu)$ or $T_n(x, \mu)$ we obtain a product of small integers.

We also remark that implicit in [3] are formulas for $Z_n(1, y, 0)$ and $Z_n(1, y, 1)$, but we have no need for these here.

3 Generating Functions for the Symmetry Classes

Now we proceed to describe what has been observed or proved concerning the symmetry classes of alternating sign matrices. There are eight symmetry classes and we discuss these in Subsections 3.1 through 3.8.

3.1 All Alternating Sign Matrices

An alternating sign matrix always has a 1 in the top row. Suppose that (a_{ij}) is an alternating sign matrix and that $a_{0s} = 1$ and that the number of entries equal to -1 is precisely r . Then we assign a weight of $x^r y^s$ to this matrix. Let $A_n(x, y)$ be the ordinary generating function for all n by n alternating sign matrices, that is, the sum of the weights of all these matrices. Then it is conjectured that

Conjecture 3.1 $A_n(x, y) = Z_{n-1}(x, y, 1)$.

3.2 Flip Symmetric Alternating Sign Matrices

This symmetry class is empty unless $n = 2m + 1$ is odd. In this case there can be no zeros in column m so that $a_{jm} = (-1)^j$. If k is the number of -1 's in the first m columns of a flip symmetric alternating sign matrix, we assign it a weight of x^k . Let $F_n(x)$ be the ordinary generating function for this symmetry class of alternating sign matrices.

Conjecture 3.2 $F_{2n+1}(x) = T_n(x, 1)$.

3.3 Invariant Under the Half-Turn in its Own Plane

We assign to alternating sign matrices in this symmetry class a weight $x^r y^s$ where $a_{0s} = 1$ and r is the number of orbits of -1 's under the action of the 2-element group generated by the half-turn. This number is half the number of -1 's in the matrix unless $n = 2m + 1$ is odd and $a_{mm} = -1$ in which case the number of -1 's is an odd number $2l + 1$ and $r = l + 1$. We denote the generating function for half-turn invariant n by n alternating sign matrices by $H_n(x, y)$.

For even size matrices we have observed the surprising formula

Conjecture 3.3 $H_{2n}(x, y) = Z_n(x, y, 0)Z_{n-1}(x, y, 1)$.

$Z_n(x, y, 0)$ is the generating function for cyclically symmetric plane partitions with Ferrer's graph contained in the box \mathcal{X}_n and $Z_{n-1}(x, y, 1)$ is conjectured to be the generating function for n by n alternating sign matrices. Thus Conjecture C1 suggests a one-to-one correspondence between half-turn symmetric $2n$ by $2n$ alternating sign matrices and the Cartesian product of the set of n by n alternating sign matrices and the cyclically symmetric plane partitions with Ferrers graph contained in the box \mathcal{X}_n .

It is previously been observed that when we substitute $x = 2$ in generating functions for alternating sign matrices, they generally become much simpler. For example, using the methods of [4] and [7] we can prove the following result.

Theorem 3.1

$$\begin{aligned}\frac{H_{4n}(2, 1)}{H_{4n-2}(2, 1)} &= 2^{2n-1} \frac{\binom{4n}{2n}}{\binom{2n}{n}} \\ \frac{H_{4n+2}(2, 1)}{H_{4n}(2, 1)} &= 2^{2n+1} \frac{\binom{4n}{2n}}{\binom{2n}{n}} \\ H_{2n+1}(2, 1) &= 2^n H_{2n}(2, 1).\end{aligned}$$

For odd size half-turn symmetric alternating sign matrices the generating functions do not seem to factor. However when we set $y = 1$, then $H_{2n+1}(x, 1)$ does seem to factor. In fact we have observed that

Conjecture 3.4 (i) $H_{4n+1}(x, 1) = R_n(x, 0)T_n(x, 1)S_{4n+1}(x)$

(ii) $H_{4n-1}(x, 1) = R_{n-1}(x, 1)T_n(x, 0)S_{4n-1}(x)$ where $S_1(x), S_3(x), \dots$ are certain polynomials.

The first few S 's are given in the table at the end of Section 4.

If we set $x = 1$ in the generating function for odd size half-turn symmetric alternating sign matrices, obtaining the polynomials $H_n(1, y)$, we have no conjecture concerning the value of the resulting function. Nevertheless these polynomials will appear later as factors of another generating function so we tabulate a few of their values in the last section.

It seems appropriate to repeat here an interesting observation from [5].

Conjecture 3.5 $A_n(3, 1) = 3^{\deg A_n(x, 1)} H_n(1, 1)$.

3.4 Equal to Transpose

We have enumerated the alternating sign matrices in this symmetry class for $n = 1, 2, \dots, 8$. Their numbers are 1, 2, 5, 16, 67, $2^4 \cdot 23$, $2 \cdot 5 \cdot 263$, $2^3 \cdot 11 \cdot 277$. Apparently these numbers do not factor into small primes, so a simple product formula seems unlikely. Of course this does not rule out other very simple formulas, but these would be more difficult to discover (let alone prove).

3.5 Invariant Under 90 Degree Rotation in its Own Plane

When n is even, each quarter of an alternating sign matrix in this symmetry class must have the sum of its entries equal to $n/4$. It follows such matrices can exist for even n only when n is actually a multiple of 4. Alternating sign matrices of all odd sizes do exist with this symmetry type. We assign to each such matrix a weight of $x^r y^s$ if $a_{0s} = 1$ and r is the number of orbits of -1 's under the action of the group generated by the quarter turn. More precisely, r is one-fourth the number of -1 's if the number of -1 's is divisible by 4 or $l + 1$ if the number of -1 's is $4l + 1$ (the only other possibility). It is clear that we must have $0 < k < n - 1$. We denote by $Q_n(x, y)$ the ordinary generating function of n by n alternating sign matrices of this type. Then it follows that $Q_n(x, y)$ must be divisible by y . We have conjectures concerning $Q_n(x, 1)$ and $Q_n(1, y)$.

Conjecture 3.6 *For $n \geq 1$*

$$\begin{aligned} Q_{4n}(1, y) &= yH_{2n}(1, y, 0)A_n(1, y)^2; \\ Q_{4n+1}(1, y) &= yH_{2n+1}(1, y)A_n(1, y)^2; \\ Q_{4n-1}(1, y) &= yH_{2n-1}(1, y)A_n(1, y)^2. \end{aligned}$$

We have verified these formulas only for $n = 1, 2, 3$ and 4 . They suggest bijections between quarter-turn symmetric alternating sign matrices and various Cartesian products with other classes of alternating sign matrices and cyclically symmetric plane partitions.

Conjecture 3.7 *There exists a sequence of polynomials $w_0(x), w_1(x), \dots$ such that $Q_{2n+1}(x, 1) = w_n(x)w_{n+1}(x)$ if n is even and $Q_{2n-1}(x, 1) = xw_n(x)w_{n+1}(x)$ if n is odd. Moreover $Q_{4n}(x, 1) = v_n(x)w_{2n}(x)$ for suitable polynomials $v_n(x)$.*

The first statement has been observed to be true for $n = 0, \dots, 7$ and the second for $n = 1, \dots, 4$. Some of the v 's and w 's are tabulated in the last section.

We have a combinatorial interpretation for $w_{2n}(x)$. Consider the cyclically symmetric plane partitions (described more fully in [6]) with Ferrers graph F contained in the box $\mathcal{X}_{2n} = [1, 2n] \times [1, 2n] \times [1, 2n]$ and which are self-complementary in the sense that for $(i, j, k) \in \mathcal{X}_{2n}$, we have $(i, j, k) \in F$ if

and only if $(2n + 1 - i, 2n + 1 - j, 2n + 1 - k) \notin F$. We say that a part a_{ij} of such a plane partition is *special* if $i \leq a_{ij} < j$. Assign a weight to such a partition to be equal to x^k where k is the number of special parts. Then the generating function for this class of functions is equal to $x^n w_{2n}(x)$ at least for $n \leq 4$. Also note that this is consistent with our previous conjecture that the number of these self-complementary cyclically symmetric plane partitions is the square of the number of n by n alternating sign matrices. (This was reported in [8].)

We note that if our other conjectures are true, then we would have $w_1(1)$, $w_3(1), \dots$, equal the numbers of half-turn symmetric alternating sign matrices of sizes 1, 3, \dots . Thus $w_{2n+1}(x)$ may be a generating function for these matrices with a suitable assignment of weights. However, we do not know of such a weight. There is a similar observation connecting $v_n(x)$ with $2n$ by $2n$ half-turn symmetric alternating sign matrices.

3.6 Invariant Under Flips in Vertical and Horizontal Axes

Alternating sign matrices with this symmetry can exist only when n is odd. When $n = 2m + 1$ we must have $a_{im} = (-1)^i$ and $a_{mj} = (-1)^j$.

For an alternating sign matrix of size $n = 2m + 1$ with this symmetry we assign a weight of x^k where k is the number of entries $a_{ij} = -1$ with $0 \leq i, j < m$. Let $P_n(x)$ be the generating function for the n by n alternating sign matrices satisfying this symmetry condition. (We use P since the axes of symmetry of the matrix look like a plus sign.)

It is possible to conjecture a formula for the P 's terms of the T 's.

Conjecture 3.8 *For $n \geq 1$*

$$\begin{aligned} P_{4n+1}(x) &= T_n(x, 1)T_n(x, 0). \\ P_{4n-1}(x) &= T_{n-1}(x, 1)T_n(x, 0). \end{aligned}$$

This has been verified only for small n .

3.7 Invariant Under Flips in Both Diagonals

For $n = 2, 4, 6, \dots, 18$ we have found that the number of such alternating sign matrices in this class is 2, 2^3 , $2^2 \cdot 13$, $8 \cdot 71$, $2^2 \cdot 2609$, $2^3 \cdot 31 \cdot 1303$,

$2^3 \cdot 17 \cdot 124021$, so apparently their number is not given as a simple product of small integers.

On the other hand, when n is odd, we can recognize a pattern, but this is given in the second table of Section 4. We have not found any interesting properties of any of the generating functions.

3.8 Invariant Under all Symmetries of the Square

We have enumerated the alternating sign matrices in this symmetry class and we find that for $n = 1, 3, 5, \dots, 17$ the numbers of such n by n alternating sign matrices are 1, 1, 1, 2, 4, 13, 46, $8 \cdot 31$, $4 \cdot 379$. Apparently these numbers do not factor into small primes, so again a simple product formula seems unlikely.

4 Tables

4.1 A Small Table of Numerical Values

In the table below the values not given have either not been computed or are too large to fit (*).

Symmetry Type								
size	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	2	0	2	2	0	0	2	0
3	7	1	3	5	1	1	3	1
4	42	0	10	16	2	0	8	0
5	429	3	25	67	3	1	15	1
6	7436	0	140	368	0	0	52	0
7	218348	26	588	2630	12	2	126	2
8	*	0	5544		40	0	568	0
9	*	646	39204		100	6	1782	4
10	*	0	622908		0	0	10436	0
11	*	45885	7422987		1225	33	42471	13
12	*	0	*		6460	0	323144	0
13	*	9304650	*		28812	286	1706562	46
14	*	0	*		0	0	*	0
15	*	*	*		1037232	4420	*	248
16	*	0	*		9779616	0	*	0
17	*	*	*	*	*	109820	*	1516

4.2 Table of Numerical Conjectures

Here the numbers of alternating sign matrices in classes 1, 2, 3, 4, 6, and 7 are denoted by A_n , F_n , H_n , Q_n , P_n and X_n . None of the results stated here

have been proved. The conjecture for the P 's is due to W. H. Mills.

$$\frac{A_{n+1}}{A_n} = \frac{\binom{3n+1}{n}}{\binom{2n}{n}}.$$

$$\frac{F_{2n+1}}{F_{2n-1}} = \frac{\binom{6n-2}{2n}}{2 \binom{4n-1}{2n}}.$$

$$\frac{H_{2n+1}}{H_{2n}} = \frac{\binom{3n}{n}}{\binom{2n}{n}}; \quad \frac{H_{2n}}{H_{2n-1}} = \frac{4 \binom{3n}{n}}{3 \binom{2n}{n}}.$$

$$Q_{4n} = H_{2n}A_n^2; \quad Q_{4n+1} = H_{2n+1}A_n^2; \quad Q_{4n-1} = H_{2n-1}A_n^2.$$

$$\frac{P_{4n+1}}{P_{4n-1}} = \frac{(3n-1) \binom{6n-3}{2n-1}}{(4n-1) \binom{4n-2}{2n-1}}; \quad \frac{P_{4n+3}}{P_{4n+1}} = \frac{(3n+1) \binom{6n}{2n}}{(4n+1) \binom{4n}{2n}}.$$

$$\frac{X_{2n+1}}{X_{2n-1}} = \frac{\binom{3n}{n}}{\binom{2n-1}{n}}.$$

4.3 Tables of Generating Functions

$$Z_1(x, y, 0) = 1 + y$$

$$Z_2(x, y, 0) = 2 + xy + 2y^2$$

$$Z_3(x, y, 0) = (4 + x) + (4x + x^2)y + (4x + x^2)y^2 + (4 + x)y^3$$

$$Z_4(x, y, 0) = 8 + 10x + 2x^2 + (12x + 15x^2 + 3x^3)y \\ + (12x + 15x^2 + 4x^3 + x^4)y^2 + \dots$$

$$Z_1(x, y, 1) = 1 + y$$

$$Z_2(x, y, 1) = 2 + (x + 2)y + 2y^2$$

$$Z_3(x, y, 1) = (6 + x) + (6 + 7x + x^2)y + (6 + 7x + x^2)y^2 + (6 + x)y^3$$

$$Z_4(x, y, 1) = (24 + 16x + 2x^2) + (24 + 52x + 26x^2 + 3x^3)y \\ + (24 + 64x + 38x^2 + 8x^3 + x^4)y^2 + \dots$$

$$T_1(x, 0) = 1$$

$$T_2(x, 0) = 1 + x$$

$$T_3(x, 0) = 1 + 5x + 4x^2 + x^3$$

$$T_4(x, 0) = 1 + 14x + 49x^2 + 62x^3 + 34x^4 + 9x^5 + x^6$$

$$T_1(x, 1) = 1$$

$$T_2(x, 1) = 2 + x$$

$$T_3(x, 1) = 6 + 13x + 6x^2 + x^3$$

$$T_4(x, 1) = 24 + 136x + 234x^2 + 176x^3 + 63x^4 + 12x^5 + x^6$$

$$R_1(x, 0) = 4 + x$$

$$R_2(x, 0) = 16 + 40x + 9x^2 + x^3$$

$$R_3(x, 0) = 64 + 560x + 1036x^2 + 629x^3 + 125x^4 + 16x^5 + x^6$$

$$R_1(x, 1) = 6 + x$$

$$R_2(x, 1) = 60 + 70x + 12x^2 + x^3$$

$$R_3(x, 1) = 840 + 3080x + 3038x^2 + 1224x^3 + 195x^4 + 20x^5 + x^6$$

$$H_1(1, y) = 1$$

$$H_3(1, y) = 1 + y + y^2$$

$$H_5(1, y) = 3 + 6y + 7y^2 + 6y^3 + 3y^4$$

$$H_7(1, y) = 25 + 75y + 123y^2 + 142y^3 + 123y^4 + 75y^5 + 25y^6$$

$$S_1(x) = 1$$

$$S_3(x) = 2 + x$$

$$S_5(x) = 2 + 3x$$

$$S_7(x) = 8 + 26x + 7x^2 + x^3$$

$$S_9(x) = 12 + 74x + 78x^2 + 31x^3 + 3x^4$$

$$w_0(x) = 1$$

$$w_1(x) = 1$$

$$w_2(x) = 1$$

$$w_3(x) = 2 + x$$

$$w_4(x) = 3 + x$$

$$w_5(x) = 4 + 14x + 6x^2 + x^3$$

$$w_6(x) = 15 + 25x + 8x^2 + x^3$$

$$w_7(x) = 8 + 88x + 222x^2 + 192x^3 + 65x^4 + 12x^5 + x^6$$

$$w_8(x) = 105 + 490x + 665x^2 + 386x^3 + 102x^4 + 15x^5 + x^6$$

$$v_1(x) = 2$$

$$v_2(x) = 4 + 6x$$

$$v_3(x) = 8 + 52x + 60x^2 + 20x^3$$

$$v_4(x) = 16 + 272x + 1212x^2 + 2000x^3 + 1470x^4 + 504x^5 + 70x^6$$

References

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